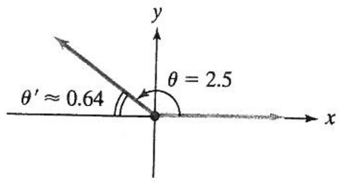


Figure 5.44

d. The angle $\theta = 2.5$ lies between $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$. This means that $\theta = 2.5$ is in quadrant II, shown in **Figure 5.44.** The reference angle is

$$\theta' = \pi - 2.5 \approx 0.64.$$

Check Point 5 Find the reference angle, θ', for each of the following angles:

a. $\theta = 210°$ **b.** $\theta = \frac{7\pi}{4}$ **c.** $\theta = -240°$ **d.** $\theta = 3.6$.

Finding reference angles for angles that are greater than 360° (2π) or less than −360° (-2π) involves using coterminal angles. We have seen that coterminal angles have the same initial and terminal sides. Recall that coterminal angles can be obtained by increasing or decreasing an angle's measure by an integer multiple of 360° or 2π.

Finding Reference Angles for Angles Greater Than 360° (2π) or Less Than −360° (-2π)

1. Find a positive angle α less than 360° or 2π that is coterminal with the given angle.
2. Draw α in standard position.
3. Use the drawing to find the reference angle for the given angle. The positive acute angle formed by the terminal side of α and the x-axis is the reference angle.

EXAMPLE 6 Finding Reference Angles

Find the reference angle for each of the following angles:

a. $\theta = 580°$ **b.** $\theta = \frac{8\pi}{3}$ **c.** $\theta = -\frac{13\pi}{6}$.

Solution

a. For a 580° angle, subtract 360° to find a positive coterminal angle less than 360°.

$$580° - 360° = 220°$$

Figure 5.45 shows $\alpha = 220°$ in standard position. Because 220° lies in quadrant III, the reference angle is

$$\alpha' = 220° - 180° = 40°.$$

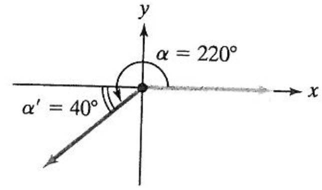


Figure 5.45

b. For an $\frac{8\pi}{3}$, or $2\frac{2}{3}\pi$, angle, subtract 2π to find a positive coterminal angle less than 2π.

$$\frac{8\pi}{3} - 2\pi = \frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$$

Figure 5.46 shows $\alpha = \frac{2\pi}{3}$ in standard position. Because $\frac{2\pi}{3}$ lies in quadrant II, the reference angle is

$$\alpha' = \pi - \frac{2\pi}{3} = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{\pi}{3}.$$

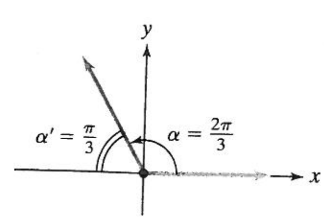


Figure 5.46

Discovery

Solve part (c) using the coterminal angle formed by adding 2π, rather than 4π, to the given angle.

c. For a $-\frac{13\pi}{6}$, or $-2\frac{1}{6}\pi$, angle, add 4π to find a positive coterminal angle less than 2π.

$$-\frac{13\pi}{6} + 4\pi = -\frac{13\pi}{6} + \frac{24\pi}{6} = \frac{11\pi}{6}$$

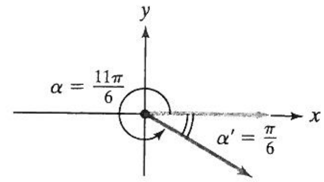


Figure 5.47

Figure 5.47 shows $\alpha = \frac{11\pi}{6}$ in standard position.

Because $\frac{11\pi}{6}$ lies in quadrant IV, the reference angle is

$$\alpha' = 2\pi - \frac{11\pi}{6} = \frac{12\pi}{6} - \frac{11\pi}{6} = \frac{\pi}{6}.$$

Check Point 6 Find the reference angle for each of the following angles:

a. $\theta = 665°$ **b.** $\theta = \frac{15\pi}{4}$ **c.** $\theta = -\frac{11\pi}{3}$.

4 Use reference angles to evaluate trigonometric functions.

Evaluating Trigonometric Functions Using Reference Angles

The way that reference angles are defined makes them useful in evaluating trigonometric functions.

Using Reference Angles to Evaluate Trigonometric Functions

The values of the trigonometric functions of a given angle, θ, are the same as the values of the trigonometric functions of the reference angle, θ', except possibly for the sign. A function value of the acute reference angle, θ', is always positive. However, the same function value for θ may be positive or negative.

For example, we can use a reference angle, θ', to obtain an exact value for tan 120°. The reference angle for $\theta = 120°$ is $\theta' = 180° - 120° = 60°$. We know the exact value of the tangent function of the reference angle: $\tan 60° = \sqrt{3}$. We also know that the value of a trigonometric function of a given angle, θ, is the same as that of its reference angle, θ', except possibly for the sign. Thus, we can conclude that tan 120° equals $-\sqrt{3}$ or $\sqrt{3}$.

What sign should we attach to $\sqrt{3}$? A 120° angle lies in quadrant II, where only the sine and cosecant are positive. Thus, the tangent function is negative for a 120° angle. Therefore,

Prefix by a negative sign to show tangent is negative in quadrant II.

$$\tan 120° = -\tan 60° = -\sqrt{3}.$$

The reference angle for 120° is 60°.

In the previous section, we used two right triangles to find exact trigonometric values of 30°, 45°, and 60°. Using a procedure similar to finding tan 120°, we can now find the exact function values of all angles for which 30°, 45°, or 60° are reference angles.

A Procedure for Using Reference Angles to Evaluate Trigonometric Functions

The value of a trigonometric function of any angle θ is found as follows:

1. Find the associated reference angle, θ', and the function value for θ'.
2. Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1.

Discovery

Draw the two right triangles involving 30°, 45°, and 60°. Indicate the length of each side. Use these lengths to verify the function values for the reference angles in the solution to Example 7.

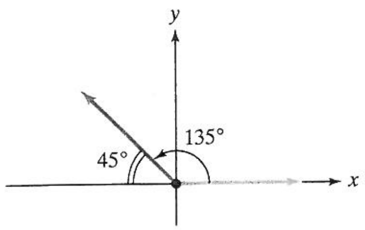


Figure 5.48 Reference angle for 135°

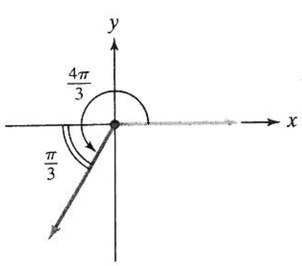


Figure 5.49 Reference angle for $\frac{4\pi}{3}$

EXAMPLE 7 Using Reference Angles to Evaluate Trigonometric Functions

Use reference angles to find the exact value of each of the following trigonometric functions:

a. $\sin 135°$ **b.** $\cos \frac{4\pi}{3}$ **c.** $\cot\left(-\frac{\pi}{3}\right)$.

Solution

a. We use our two-step procedure to find $\sin 135°$.

Step 1 Find the reference angle, θ', and $\sin \theta'$. Figure 5.48 shows 135° lies in quadrant II. The reference angle is

$$\theta' = 180° - 135° = 45°.$$

The function value for the reference angle is $\sin 45° = \frac{\sqrt{2}}{2}$.

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The angle $\theta = 135°$ lies in quadrant II. Because the sine is positive in quadrant II, we put a + sign before the function value of the reference angle. Thus,

The sine is positive in quadrant II.

$$\sin 135° = +\sin 45° = \frac{\sqrt{2}}{2}.$$

The reference angle for 135° is 45°.

b. We use our two-step procedure to find $\cos \frac{4\pi}{3}$.

Step 1 Find the reference angle, θ', and $\cos \theta'$. Figure 5.49 shows that $\theta = \frac{4\pi}{3}$ lies in quadrant III. The reference angle is

$$\theta' = \frac{4\pi}{3} - \pi = \frac{4\pi}{3} - \frac{3\pi}{3} = \frac{\pi}{3}.$$

The function value for the reference angle is

$$\cos \frac{\pi}{3} = \frac{1}{2}.$$

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The angle $\theta = \frac{4\pi}{3}$ lies in quadrant III. Because only the tangent and cotangent are positive in quadrant III, the cosine is negative in this quadrant. We put a − sign before the function value of the reference angle. Thus,

The cosine is negative in quadrant III.

$$\cos \frac{4\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}.$$

The reference angle for $\frac{4\pi}{3}$ is $\frac{\pi}{3}$.

c. We use our two-step procedure to find $\cot\left(-\frac{\pi}{3}\right)$.

Step 1 Find the reference angle, θ', and $\cot\theta'$. **Figure 5.50** shows that $\theta = -\frac{\pi}{3}$ lies in quadrant IV. The reference angle is $\theta' = \frac{\pi}{3}$. The function value for the reference angle is $\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$.

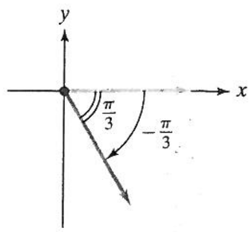


Figure 5.50 Reference angle for $-\frac{\pi}{3}$

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The angle $\theta = -\frac{\pi}{3}$ lies in quadrant IV. Because only the cosine and secant are positive in quadrant IV, the cotangent is negative in this quadrant. We put a − sign before the function value of the reference angle. Thus,

The cotangent is negative in quadrant IV.

$$\cot\left(-\frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -\frac{\sqrt{3}}{3}.$$

The reference angle for $-\frac{\pi}{3}$ is $\frac{\pi}{3}$.

Check Point 7 Use reference angles to find the exact value of the following trigonometric functions:

a. $\sin 300°$ **b.** $\tan\frac{5\pi}{4}$ **c.** $\sec\left(-\frac{\pi}{6}\right)$.

In our final example, we use positive coterminal angles less than 2π to find the reference angles.

EXAMPLE 8 Using Reference Angles to Evaluate Trigonometric Functions

Use reference angles to find the exact value of each of the following trigonometric functions:

a. $\tan\frac{14\pi}{3}$ **b.** $\sec\left(-\frac{17\pi}{4}\right)$.

Solution

a. We use our two-step procedure to find $\tan\frac{14\pi}{3}$.

Step 1 Find the reference angle, θ', and $\tan\theta'$. Because the given angle, $\frac{14\pi}{3}$ or $4\frac{2}{3}\pi$, exceeds 2π, subtract 4π to find a positive coterminal angle less than 2π.

$$\theta = \frac{14\pi}{3} - 4\pi = \frac{14\pi}{3} - \frac{12\pi}{3} = \frac{2\pi}{3}$$

Figure 5.51 shows $\theta = \frac{2\pi}{3}$ in standard position. The angle lies in quadrant II. The reference angle is

$$\theta' = \pi - \frac{2\pi}{3} = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{\pi}{3}.$$

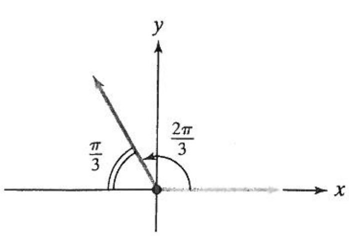


Figure 5.51 Reference angle for $\frac{2\pi}{3}$

The function value for the reference angle is $\tan\frac{\pi}{3} = \sqrt{3}$.

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The coterminal angle $\theta = \frac{2\pi}{3}$ lies in quadrant II. Because the tangent is negative in quadrant II, we put a $-$ sign before the function value of the reference angle. Thus,

$$\tan\frac{14\pi}{3} = \tan\frac{2\pi}{3} = \underbrace{-}_{\text{The tangent is negative in quadrant II.}}\tan\underbrace{\frac{\pi}{3}}_{\text{The reference angle for } \frac{2\pi}{3} \text{ is } \frac{\pi}{3}.} = -\sqrt{3}.$$

b. We use our two-step procedure to find $\sec\left(-\frac{17\pi}{4}\right)$.

Step 1 Find the reference angle, θ', and $\sec\theta'$. Because the given angle, $-\frac{17\pi}{4}$ or $-4\frac{1}{4}\pi$, is less than -2π, add 6π (three multiples of 2π) to find a positive coterminal angle less than 2π.

$$\theta = -\frac{17\pi}{4} + 6\pi = -\frac{17\pi}{4} + \frac{24\pi}{4} = \frac{7\pi}{4}$$

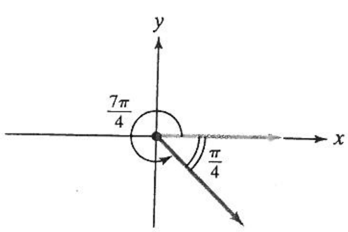


Figure 5.52 Reference angle for $\frac{7\pi}{4}$

Figure 5.52 shows $\theta = \frac{7\pi}{4}$ in standard position. The angle lies in quadrant IV. The reference angle is

$$\theta' = 2\pi - \frac{7\pi}{4} = \frac{8\pi}{4} - \frac{7\pi}{4} = \frac{\pi}{4}.$$

The function value for the reference angle is $\sec\frac{\pi}{4} = \sqrt{2}$.

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The coterminal angle $\theta = \frac{7\pi}{4}$ lies in quadrant IV. Because the secant is positive in quadrant IV, we put a $+$ sign before the function value of the reference angle. Thus,

$$\sec\left(-\frac{17\pi}{4}\right) = \sec\frac{7\pi}{4} = \underbrace{+}_{\text{The secant is positive in quadrant IV.}}\sec\underbrace{\frac{\pi}{4}}_{\text{The reference angle for } \frac{7\pi}{4} \text{ is } \frac{\pi}{4}.} = \sqrt{2}.$$

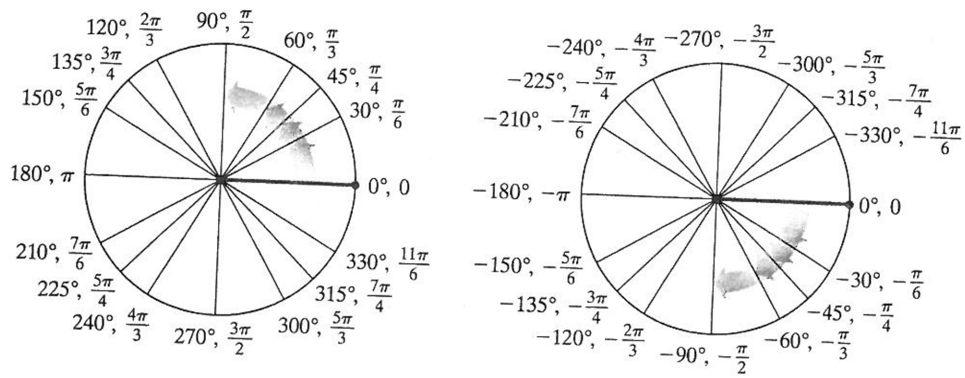
Check Point 8 Use reference angles to find the exact value of each of the following trigonometric functions:

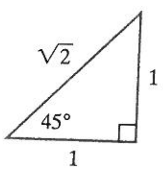
a. $\cos\frac{17\pi}{6}$ **b.** $\sin\left(-\frac{22\pi}{3}\right)$.

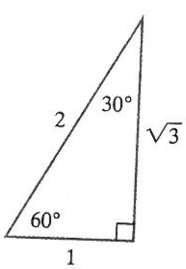
Study Tip

Evaluating trigonometric functions like those in Example 8 and Check Point 8 involves using a number of concepts, including finding coterminal angles and reference angles, locating special angles, determining the signs of trigonometric functions in specific quadrants, and finding the trigonometric functions of special angles $\left(30° = \frac{\pi}{6}, 45° = \frac{\pi}{4}, \text{ and } 60° = \frac{\pi}{3}\right)$. To be successful in trigonometry, it is often necessary to connect concepts. Here's an early reference sheet showing some of the concepts you should have at your fingertips (or memorized).

Degree and Radian Measures of Special and Quadrantal Angles



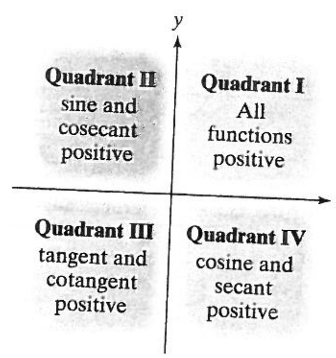




Special Right Triangles and Trigonometric Functions of Special Angles

θ	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Signs of the Trigonometric Functions



Trigonometric Functions of Quadrantal Angles

θ	$0° = 0$	$90° = \frac{\pi}{2}$	$180° = \pi$	$270° = \frac{3\pi}{2}$
$\sin\theta$	0	1	0	−1
$\cos\theta$	1	0	−1	0
$\tan\theta$	0	undefined	0	undefined

Using Reference Angles to Evaluate Trigonometric Functions

$\sin\theta = \boxed{}\ \sin\theta'$

$\cos\theta = \boxed{}\ \cos\theta'$

$\tan\theta = \boxed{}\ \tan\theta'$

+ or − in $\boxed{}$ determined by the quadrant in which θ lies and the sign of the function in that quadrant.